Mathematical Solution to Kinematic Waves

The solution to the Kinematic wave equations introduced in previous chapter is any arbitrary function of the form $n(x,t) = f(ax + bvt)$.

However we need to choose our function carefully. Fully knowing that the expected solution is a wave,

By examination the wave length is $L + D$ and wave number (number of waves in a period) is $\frac{5280}{L+D}$ which is the density

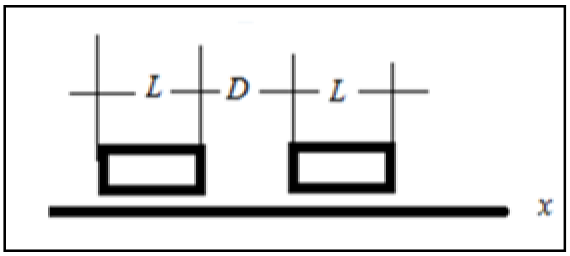

Figure 1

Thus we may write

$$f(x,t) = \frac{5280x}{L+D} + \frac{5280vt}{L+D}$$

$$f(x,t) = n(x + vt)$$

Recognizing the product of $v_f n = Q$ $(v_f\ is\ MPH)$, similarly the product of $\frac{L}{\tau_s} n = \bar{c}q$, we write the solution to the first and the second differential equation as:

$$nx + Q\,t \qquad \forall \;\; n < n_c$$

And

$$nx - \bar{c}qt \qquad \forall \;\; n > n_c$$

Where the first term describes the shape density shape and second term describes transport of it[1].

The above equations simply define the argument of the wave function. To construct the function itself we may write a Heaviside boxcar function ב(x) represented by the Hebrew letter beth with the above argument to represent a group of vehicles with a given speed and density. The number of vehicles in the group N is the length defined by the multiples of L+D such that $|(x_e - x_b| = N(L + D)$

$$ב(x, n, t) = [H(n(x - x_b - v_f t)) - H(n(x - x_e - v_f t))]$$

The above function represents a rectangular shape starting at x_b and ending at x_e travelling at v_f

Transporting the density n ant the flow $v_f n = Q$

To test the validity of the above function, first we take the derivative of it with respect to time *t* and with respect to distance *x*

$$\frac{\partial ב(x, n, t)}{\partial x} = n\delta(n(x - x_b - v_f t)) - n\delta(n(x - x_e - v_f t))$$

And

$$\frac{\partial ב(x, n, t)}{\partial t} = -nv_f\delta(n(x - x_b - v_f t)) + nv_f\delta(n(x - x_e - v_f t))$$

Adding the two together

$$\frac{\partial ב(x, n, t)}{\partial x} + v_f \frac{\partial ב(x, n, t)}{\partial t} = 0 \ \ (Q.E.D)$$

The number of vehicles that are accumulated in the backup in the time t_{acc} is then

$$\frac{qt_{acc}}{3600}$$

And the time it will take to disperse the accumulated backed up traffic is then

$$t_{disperssion} = \frac{qt_{acc}}{3600}\tau$$

As expected the time of dispersion is longer than the time of accumulation by the factor of τ.

To construct the Heaviside function one may employ the sign function $sgn(x)$

Where $sgn(x)$is the sign function defined as

$$sgn(x) = \frac{x}{|x|} \qquad \forall \ x \neq 0$$

$$ב(x) = \frac{1}{2}[sgn(x + a) - sgn(x - b)]$$

Then one can make it travel to the right

$$ב(x) = \frac{1}{2}[sgn(x + a - vt) - sgn(x - b - vt)]$$

We have formed a box car function representing 4 vehicles with the density of 30 vehicles per mile in Figure 5.

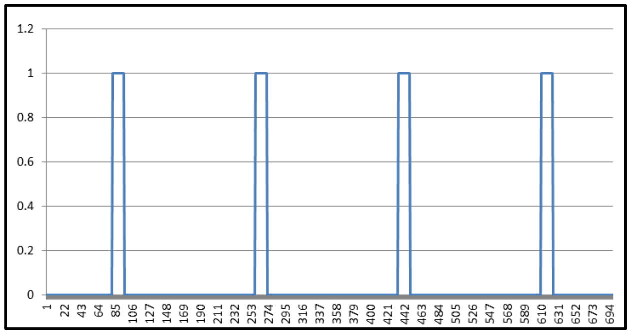

Figure 2

Merging of two streams of vehicles is done via “Exclusive Or” operation.