Essentials of Traffic Flow Theory

Anush Badii

California Department of Transportation

August 2016

Traffic either flows freely or congestively depending on the density of vehicles on the roadway. The density of the vehicle although is measured as the number of vehicles per mile is simply determined by the distance between two consecutive vehicles. As such the density is highly variable if one is looking at any stretch of roadway. For our purposes we define the density n as

$$n = \frac{5280}{L + D}$$

Where 5280 is the number of feet per mile and L is typical length of a vehicle measured in feet and D is the distance between two consecutive vehicles measured in feet.

As such, the differential of flow Q measured as the number of vehicles passing a marker per hour with respect to the density is

$$\frac{dQ}{dn} = V_s$$

Where V_s is the posted speed limit. Integrating the above and requiring that the flow will be zero if the density is zero we, arrive at

$$Q = nV_s$$

The equation for free flowing traffic. The congestive traffic flow occurs when the density of the flow exceeds a critical value ie: $n > n_{critical}$

Then the flow can be described as

$$\frac{dQ}{dn} = -\frac{L}{\tau\tilde{c}}$$

Integrating the above we may write

$$Q = -\frac{L}{\tau\tilde{c}}n + C$$

The integration constant C will be

$$\frac{5280}{\tau\tilde{c}}$$

Where τ is the headway ie the time of impact between two vehicles if the one in the front were to come to a sudden stop!. The value of the headway in free flow can be anything, however in congested flow it rapidly converges to a constant that varies with regional driving habits. In Southern California the

headway is measured at 1.75 seconds and in Seoul South Korea it is pegged at 2.5 seconds. The other constant $\tilde{c}$ conversion factor from feet per second to miles per hour which is equivalent to $\frac{5280.}{3600}$

At this point we can determine the value of $n_{critical}$ so that we can determine what flow regime the roadway is operating in.

n_c may be computed from the intersection of the two flow lines ie

$$Q = -\frac{L}{\tau\tilde{c}}n + \frac{5280}{\tau\tilde{c}}$$

And

$$Q = nV_s$$

Equating the two sides together and solving for n we obtain

$$n_c = \frac{5280}{V_s\tau\tilde{c} + L}$$

The only remaining mystery is the integration constant $\frac{5280}{\tau\tilde{c}}$ that we introduced without proof.

From the definition of density

$$n = \frac{5280}{L + D}$$

We solve for the distance

$$D = \frac{5280 - nL}{n}$$

Dividing by the speed we arrive at the equation for the headway

$$\frac{D}{v} = \tau = \frac{5280 - nL}{nv}$$

Rewriting the above for the speed and converting it into miles/hour

$$v = \frac{5280 - nL}{n\tau\tilde{c}}$$

Multiplying by the density to get the flow

$$Q = nv = n\frac{5280 - nL}{n\tau\tilde{c}} = \frac{5280}{\tau\tilde{c}} - \frac{nL}{\tau\tilde{c}}$$

If the two equations below represent the same flow then the constant of integration C in

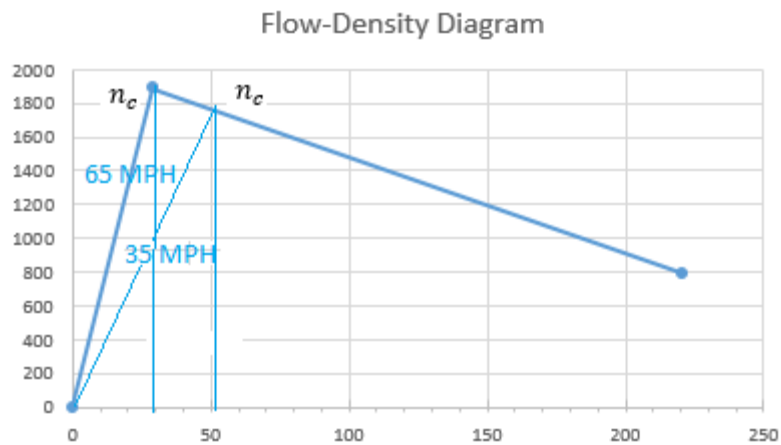
$$Q = -\frac{L}{\tau\tilde{c}}n + C$$

$$Q = \frac{5280}{\tau\tilde{c}} - \frac{nL}{\tau\tilde{c}}$$

Must by necessity be equal to

$$C = \frac{5280}{\tau\tilde{c}}$$

$\frac{5280}{\tau\tilde{c}}$ Represents the maximum flow capacity of a lane. The above equation is valid for any columnar motion be it a column of ants or vehicles.



Any transition in flow rate on the roadway is subsequent to the hourly flow rates of vehicles entering q_{ramp} entering the roadway or leaving$-q_{ramp}$the roadway barring any stoppage or lane modification.

Without proof we assert that

$$q_{ramp} = N\frac{(5280 - n_{upstream}L)}{n_{upstream}L}(Q_{upstream} - Q_{merge})$$

Where N is the number of roadway lanes.

Which will simultaneously affect both the density n_{merge}

$$q_{ramp} = N(n_{merge}v_{upstream} - Q_{upstream})$$

And speed v_{merge} according to

$$q_{ramp} = \frac{NQ_{upstream}(-Q_{upstream}L + 5280v_{upstream})(v_{upstream} - v_{merge})}{-v_{merge}Q_{upstream}L + 5280v_{merge}v_{upstream} + Lv_{upstream}Q_{upstream}}$$

The above equations provide a methodology with which the discharge of the ramp may be controlled to attain a desirable outcome in speed, flow rate or operational density of the roadway.

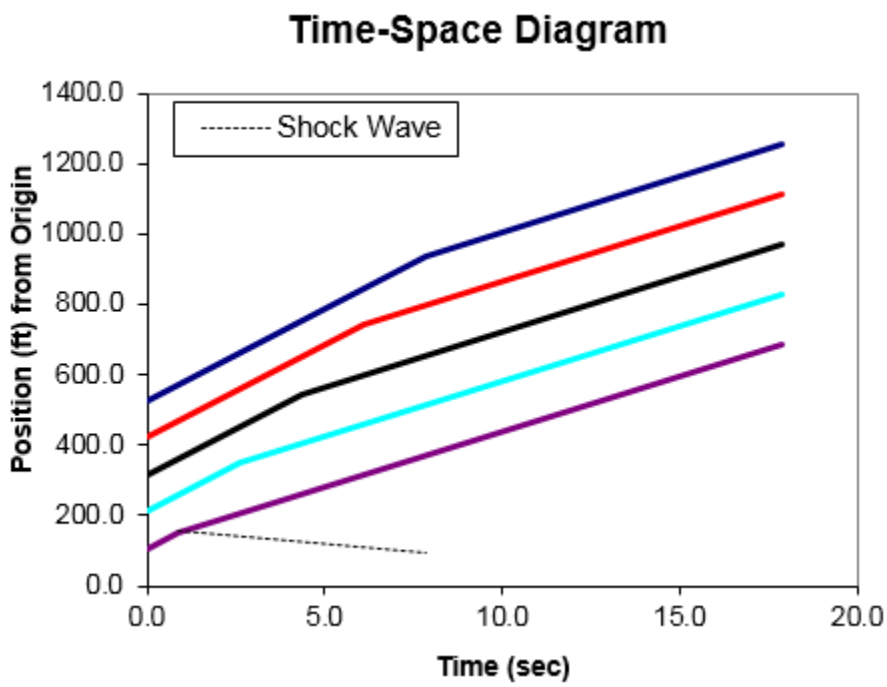
The figure $-\frac{L}{\tau\tilde{c}}n$ represents the backed up flow rate $Q_{backedup}$ where when subtracted from the maximum capacity $\frac{5280}{\tau\tilde{c}}$ will result in the manifest flow rate.

It is interesting to point out the actual backed up flow is the difference of number of vehicles involved in the back up as $Q_{Actual\ backedup} = n_c V_s - \frac{L}{\tau\tilde{c}}n$ since the maximum capacity $\frac{5280}{\tau\tilde{c}}$ cannot be ever achieved with speed limit restrictions.

The actual number of vehicles delayed is thus

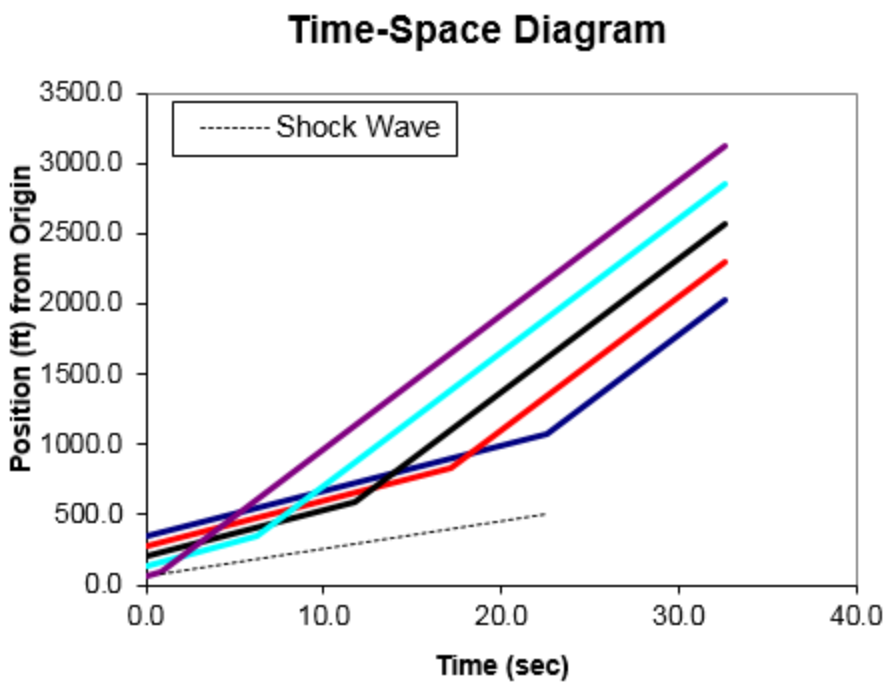
$$\aleph_{delayed} = Q_{Actualbackedup}\, t$$

Where t is measured in hours from the onset of $n > n_c$ per lane. The factor $-\frac{L}{\tau\tilde{c}}$ can be viewed as the speed of a shock wave travelling backwards that adjusts the speed of incoming vehicles to correspond to the speed of the density of vehicles ahead of them. The diagram below elucidates the transition from density of 50 to 75 vehicles per mile.



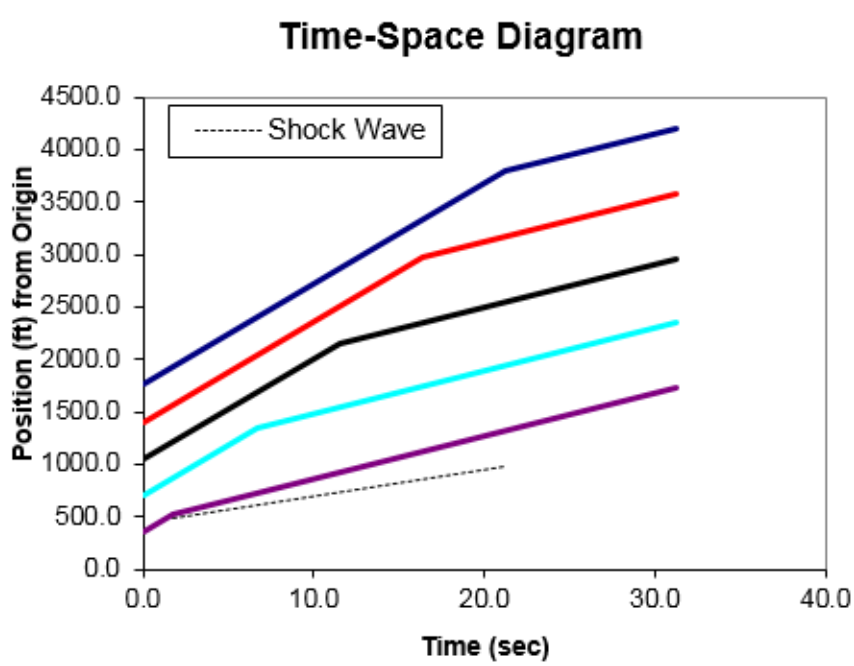
The shock wave travels upward (backwards into the traffic) at the speed of $-\frac{L}{\tau\tilde{c}}$, . The change in slope of the trajectory of each vehicle occurs when it encounters the shock wave. The backward travelling shockwave occurs in congested traffic only and forces all traffic to flow at the density of the slowest traffic along the roadway.

If there is a sudden transition from uncongested density to congested density or vice versa , the speed of the wave may not be $-\frac{L}{\tau\tilde{c}}$ and in fact may be positive $\frac{dQ}{dn} > 0$. The figure below elucidates a rapid transition from the congested flow regime to uncongested regime.



Which explains a sudden decompression such as vehicles approaching a stop light that has turned green.

The reverse is also true when a forward moving shock wave rapidly compresses the traffic flow as in



Platooning. As a part of traffic management platooning is controlled by utilizing ramp meter.

Traffic Management and the Inverse Functions

The practitioners of traffic engineering may be more interested to discover how they can influence the outcome of traffic flow through management of access to the on-ramps. To accomplish this one needs to predicate on the control of one of the traffic flow variables. The following serves as proof of the assertions earlier.

The traffic engineering practitioner may choose to control the density, flow volume, or the queue on the ramp. In the following discussions we have provided a prescription for control of these variables via the inverse functions.

$$n_{merge} = n_{ramp} + n_{upstream}$$

Substituting for their values

$$n_{merge} = \frac{q_{ramp}}{v_{upstream}} + \frac{Q_{upstream}}{v_{upstream}}$$

The merge density of the traffic on the main lanes after a flow from the ramp to an N lane facility is

$$n_{merge} = \frac{q_{ramp}}{N v_{upstream}} + \frac{Q_{upstream}}{v_{upstream}}$$

If controlling the density is a goal we utilize the above equation by asserting $n_{merge} = n_{goal}$

$$n_{goal} = \frac{q_{ramp}}{N v_{upstream}} + \frac{Q_{upstream}}{v_{upstream}}$$

$$q_{ramp} = N\left(n_{goal} v_{upstream} - Q_{upstream}\right)$$

Where N is the number of the main lanes. A negative value describes a flow rate that needs to be redirected out of freeway to accomplish the goal. As such predicating on density is not a good choice for traffic control since we have no control over the number of vehicles that need to be redirected out of the system.

A better choice to maintain a sustainable flow gradient to optimize travel time is the control of flow on the main lanes for a sustainable flow gradient. The derivation is tedious but straight forward.

The speed of the merge may be described by the equation of merge

$$v_{merge} = v_{upstream} \frac{D_{merge}}{D_{upstream}}$$

We may rewrite the above in terms of respective density

$$v_{merge} = v_{upstream} \frac{\dfrac{5280 - \left[\dfrac{q_{ramp}}{N v_{upstream}} + \dfrac{Q_{upstream}}{v_{upstream}}\right] L}{\dfrac{q_{ramp}}{N v_{upstream}} + \dfrac{Q_{upstream}}{v_{upstream}}}}{\dfrac{5280 - \left[\dfrac{Q_{upstream}}{v_{upstream}}\right] L}{\dfrac{Q_{upstream}}{v_{upstream}}}}$$

Solving the above for q_{ramp},we have a prescription for desired speed after the merge v_{merge}

$$q_{ramp} = \frac{N Q_{upstream}(-Q_{upstream} L + 5280 v_{upstream})(v_{upstream} - v_{merge})}{-v_{merge} Q_{upstream} L + 5280 v_{merge} v_{upstream} + L v_{upstream} Q_{upstream}}$$

And multiplying v_{merge} by

$$n_{merge} = \frac{q_{ramp}}{N v_{upstream}} + \frac{Q_{upstream}}{v_{upstream}}$$

and a bit more of algebraic manipulation we arrive at

$$Q_{merge} = Q_{upstream} - \frac{n_{upstream} L q_{ramp}}{N\left(5280 - n_{upstream} L\right)}$$

Where N is the number of main lanes.

Rewriting the above

$$q_{ramp} = N \frac{\left(5280 - n_{upstream} L\right)}{n_{upstream} L} (Q_{upstream} - Q_{merge})$$

$$q_{ramp} = N \frac{\left(5280 - n_{upstream} L\right)}{n_{upstream} L} (\Delta Q)$$